# Porous medium model: an algebraic perspective and the Fick's law

Renato De Paula and Chiara Franceschini

**Abstract** In this work, we study the porous medium model (PMM), an interacting particle system with nearest neighbor interactions of particles under some constraints. First, we consider the discrete space $\{1,\ldots,n-1\}$ with additional Glauber dynamics acting respectively on sites 0 and $n$. We assume the hydrodynamic limit (proved in a companion paper [3]) and we prove that the Fick's law holds. Moreover, we review how to construct a self-duality relation starting from the reversible measure of the process. Following this method, we show a self-duality result for the process without reservoirs, which is found inspired by its description via the Lie algebra $\mathfrak{su}(2)$.

**Key words:** Porous medium model, Hydrodynamic limit, Fick's law, Porous medium equation, Boundary conditions, Stochastic duality, Lie algebra $\mathfrak{su}(2)$

## 1 Introduction

One of the major problems in non-equilibrium Statistical Mechanics is the study of scaling limits of interacting particle systems (IPS). In particular, the derivation of macroscopic partial differential equations (PDE's) from microscopic systems, known in the literature as hydrodynamic limit, see [6] for a review. In recent years, a lot of attention has been devoted to the study of the asymptotic behavior of mi-

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croscopic systems coupled with reservoirs, which bring up boundary conditions to the associated hydrodynamic equation. In the case of microscopic systems with independent particles, we usually have linear hydrodynamic equations, as in [1]. Otherwise, which is the case of the PMM, we usually have nonlinear hydrodynamic equations, as in [12].

The PMM is a special case of the KLS model introduced by Katz, Lebowitz, and Spohn [17] where particles only hope randomly with no bias to nearest neighbors sites with rates depending on the occupation of the nearest and next-nearest-neighbor sites. See also [15], where, using their notation the PMM corresponds to the case which has hopping rates given by $\delta = -1$ and $\varepsilon = 0$. More recently in [12] the authors derived for the first time the porous medium equation (PME) considering discrete occupational variables. In addition, in [1], the authors studied the simple symmetric exclusion process (SSEP) combined with a Glauber dynamics, that they called "slow reservoirs", which has the heat equation with Dirichlet, Neumann, and Robin boundary conditions as hydrodynamic equations. In [3], in order to study nonlinear versions of the hydrodynamic equations obtained in [1], the authors considered the PMM with a microscopic perturbation and slow reservoirs. Thus, they derived for the first time the PME with similar boundary conditions as [1].

In this paper, we will work with two versions of the PMM. In the first part of the paper, our microscopic system of interest will be the perturbed PMM with slow reservoirs evolving in the discrete space $\{1,\dots,n-1\}$, as in [3]. The perturbation is necessary in order to assume the validity of the hydrodynamic limit through the Entropy method of Guo, Papanicolau, and Varadhan [14]. The name "slow" comes from the fact that we have a parameter $\theta \in [0,+\infty)$ which regulates the reservoirs' strength. In the second part, our microscopic system will be the PMM without perturbations and without reservoirs, evolving in the one-dimensional discrete torus $\mathbb{T}_n = \{0,1,\dots,n-1\}$. The aforementioned open models belong to the class of diffusive systems. To illustrate, consider a finite volume containing interacting particles coupled with opposite reservoirs, one at the left boundary and another at the right boundary, both having a different density of particles. In this situation, one predicts a net flux of particles from the reservoir with a higher density to the reservoir with a lower density. Therefore, after some initial transitions, we expect a non-equilibrium steady state to arise in the system, i.e., a state with a nonzero flux of particles that is constant in space and time.

The aim of this paper is to examine some questions that arise when studying diffusive systems out of equilibrium. We focus on Fick's law of diffusion, and on the self-duality for the generator of the PMM in the one-dimensional discrete torus.

The first result of this paper regards the Fick's law of diffusion, derived by Adolf Fick in [9], that says that the rate of the flux of particles is proportional to the density gradient. Although the authors in [3] studied scaling limits for the empirical density of the perturbed PMM with slow reservoirs, in the first part of this paper we study scaling limits for the empirical currents of this model. The motivation comes from [2], in which the authors derived the large deviation principle for the empirical currents of the SSEP in the domain $\{-n,n\}$ with creation and annihilation of particles in the bulk and a Glauber dynamics at the boundaries.

As mentioned above the PMM has the porous medium equation as hydrodynamic equation. It is a partial differential equation that can be seen in dimension one as

$$\partial_t \rho = \Delta \rho^M, \quad M > 1. \tag{1}$$

It is a nonlinear diffusion equation that can be written in the divergence form as

$$\partial_t \rho = \nabla(D(\rho)\nabla\rho),$$

where $\rho = \rho(t,u)$ is a scalar function, which in this paper, denotes the macroscopic density of particles in $u \in [0,1]$ at time $t > 0$, and $D(\rho) = M\rho^{M-1}$ is the diffusion coefficient. The equation is parabolic at the points where $\rho \neq 0$, but it changes its character at the level $\rho = 0$, since $D(\rho)$ vanishes as $\rho \to 0$. See [22] for more details. In this work, for simplicity of the presentation, we will consider the case $M = 2$, so that the hydrodynamic equations studied here are the same that the ones in [3]. These equations will be the PME with boundary conditions depending on the parameter $\theta$, which, as mentioned above, regulates the reservoirs' strengths. For $0 \leq \theta < 1$, we have the PME with Dirichlet boundary conditions (19); For $\theta = 1$, the boundary dynamics is slowed enough so the boundary conditions of Dirichlet type are replaced by a type of Robin boundary conditions (20); Finally, for $\theta > 1$, the boundary is sufficiently slowed so that the Robin boundary conditions are replaced by Neumann boundary conditions.
Therefore, with the notations above, we can write the Fick's law in dimension one as

$$J = -D(\rho)\nabla\rho,$$

where $J$ is the diffusion flux. We stress that throughout the first part of the paper we are assuming the validity of the hydrodynamic limit for the perturbed PMM with slow reservoirs, which was proved in [3]. Thus, for the convenience of the reader, we repeat the relevant material from [3] without proofs, thus making our exposition self-contained.
The second result of this paper regards self-duality for the generator of the PMM whose dynamics take place in the discrete one-dimensional torus, $\mathbb{T}_n = \{0,1,\ldots,n-1\}$. Duality, first introduced by Liggett in [20], is a powerful and rare tool to deal with Markov processes and, in particular, interacting particle systems.
Duality relations allow us to connect two Markov processes via a duality function; such functions are observables in terms of both processes whose expectations satisfy a specific relation. We speak of self-duality if the two Markov processes are two independent copies of the same process. The usefulness of (self-)duality is due to the fact that the dual process may be easier to deal with than the initial process. Duality plays a role in non-equilibrium statistical mechanics: a microscopic knowledge of particle systems can only be reached for a particular class of model - known as exactly solvable models.
Unfortunately, these models, for which we can analytically compute profiles and covariances in non-equilibrium settings, are rare and it turns out that they exhibit a self-duality property. Two examples are the SSEP and the KMP model, which

describes a system of one-dimensional oscillators that redistribute energy among nearest neighbors [19]. In this context, the self-duality relations can be used to infer information regarding the $n-$point correlation functions using information (when available) on $n$ dual particles. The literature so far has been concentrated on several IPS and diffusion processes (see e.g. [5]): among the others the SSEP, of which the PMM share several aspects but it is, however, more complicated.
It is not hard to see that from the PMM one can always retrieve results regarding the SSEP: this is done by setting $M = 1$ in the hydrodynamic equation above and microscopically by setting the exchange rate by 1. For the SSEP a meaningful duality relation is well-known to exist and it is related to the fact that the stationary correlation functions satisfy linear difference equations not involving correlation of higher orders [21], see also Remark 11 below. In this case, an explicit expression of such correlations is known and can be used in the study, for example, of the hydrostatic equation for which an upper bound on the two-points correlation function is needed with the aim of knowing the decreasing rate of convergence to zero. This preliminary study of duality for PMM is motivated by the fact that the hydrostatic limit is still an open problem. As mentioned above, at the microscopic level, PMM has kinetic constraints given by the configuration while macroscopically one derives a nonlinear PDE as hydrodynamic limit. One of the main issues is that the equations given by the (same time) correlations are not closed in the sense that at each step the degree is increased by one; for this reason, finding a meaningful duality relation is far from trivial. On these grounds, this second part has a more investigative approach as, so far, no duality relations are known, for systems with such features.

### *1.1 Organization of the paper*

Our presentation is divided into two sections: one to derive the Fick's law for our model under the hypothesis of hydrodynamic limit and another to provide an algebraic perspective of the model. We have divided them in such a way that the reader can read Sects. 2 and 3 separately. In Sects. 2.1 and 2.2, we look more closely at the instantaneous and integrated currents of the model. In Sect. 2.3, we defined the empirical measures associated with these currents. In Sect 2.4, we present the notion of weak solution of the PME with different boundary conditions, to finally prove, in Sect. 2.5, that the Fick's law holds. Sect. 3 starts with a review of duality theory in the context of interacting particle systems. In Sect. 3.1, we explain the algebraic approach to duality by introducing the Lie algebra $\mathfrak{su}(2)$ and we describe how, starting from the reversible measure of the process one can find some duality relations. In Sect. 3.2, we describe the bulk of our model via the generators of the $\mathfrak{su}(2)$ algebra and, lastly, in Sect. 3.3 we show how two different self-duality functions can be constructed. We end this first section by describing the model in two different settings: first,we illustrate its bulk dynamics, which is common to both Sects. 2 and 3; then, we have a subsection to develop the perturbed dynamics in an open setting, by superposing it with a SSEP dynamics and adding two external reservoirs at the boundary, needed for Section 2 only.

## 1.2 The model

The bulk of the PMM is a continuous time Markov process where particles jump under the exclusion rule to nearest neighbor sites according to the state of the process. However, there are some constraints to take into consideration. Our discrete space is $\Sigma_n := \{1,\dots,n-1\}$. Suppose that a particle at the site $x$ wants to perform a jump to the site $x+1$: the jump with rate 1 is allowed only if there is a particle at the site $x-1$ or at the site $x+2$. If these sites are empty, then the particle at site $x$ cannot jump and we called it blocked; if both the aforementioned sites are occupied, then the particle performs the jump with rate 2. Due to the constraints of the model's rates, the PMM has configurations that do not evolve under the dynamics of the model, the so-called blocked configurations. The construction of the process is done in the following way. For each $x \in \Sigma_n$, the occupation variable $\eta(x)$ denotes the number of particles at site $x$, where $\eta(x) = 0$ (resp. $\eta(x) = 1$) stands for empty (resp. occupied) site, which makes our state space $\Omega_n := \{0,1\}^{\Sigma_n}$. We denote by $\eta \in \Omega_n$ the configuration of particles. To each bond of the bulk $\{x,x+1\}$ with $x = 1\dots,n-2$, we associate three Poisson clocks with a parameter depending on the exclusion rule and on the constraints of the process, which are represented by the following Poisson processes: $N^{x-1}_{x,x+1}(t)$ and $N^{x-2}_{x,x+1}(t)$ with parameter 1, while $N_{x,x+1}(t)$ with parameter 2. The PMM generator describes the evolution of the process and it acts on functions $f : \Omega_n \to \mathbb{R}$ as

$$L_P f(\eta) = \sum_{x=1}^{n-2} c_{x,x+1}(\eta)\,\{a_{x,x+1}(\eta) + a_{x+1,x}(\eta)\}\,(f(\eta^{x,x+1}) - f(\eta)) \tag{2}$$

where

$$c_{x,x+1}(\eta) = \eta(x-1) + \eta(x+2), \tag{3}$$

$$a_{x,x+1}(\eta) = \eta(x)(1-\eta(x+1)), \tag{4}$$

are the exchange rates, while the exchange configuration $\eta^{x,y}$ is given by

$$\eta^{x,y}(z) = \begin{cases} \eta(z), \; z \neq x,y, \\ \eta(y), \; z = x, \\ \eta(x), \; z = y. \end{cases}$$

Notice that $L_P$ conserves the total number of particles.

### 1.2.1 The open PMM

Let us now describe the open dynamics of the perturbed PMM with slow reservoirs. Fix the following real numbers: $1 < a < 2$, $\theta \geq 0$, $m > 0$, and $\alpha, \beta \in (0,1)$. Let $n \geq 1$ be a scaling parameter. The particles are distributed on the points of the discrete space $\Sigma_n$. We artificially add two external sites 0 and $n$ where particles can be inserted or removed from the bulk with some rates defined below. We associate two Poisson clocks at the bonds $\{0,1\}$ and $\{n-1,n\}$ in the following way: $N_{0,1}(t)$ (resp. $N_{n,n-1}(t)$) with parameter $m\alpha n^{-\theta}$ (resp. $m\beta n^{-\theta}$) and $N_{1,0}(t)$ (resp. $N_{n-1,n}(t)$) with parameter $m(1-\alpha)n^{-\theta}$ (resp. $m(1-\beta)n^{-\theta}$). The Poisson processes associated to a bulk bond are now affected by the superposed SSEP dynamics, thus we need to add a factor of $n^{a-2}$ to their parameters. We stress that all of these Poisson processes are independent and throughout the text we use the convention

$$\eta(0) = \alpha, \quad \eta(n) = \beta.$$

For a description of the dynamics of the model, see Figure 1.2.1.
The perturbed PMM with slow reservoirs is a continuous time Markov process $\{\eta_t\}_{t\geq 0}$ on $\Omega_n = \{0,1\}^{\Sigma_n}$. It can be fully characterized by the infinitesimal generator $L_n$ given by

$$L_n = L_P + n^{a-2} L_S + L_\alpha + L_\beta, \tag{5}$$

where $L_P$ is the bulk generator of the PMM defined in equation (2), while $L_S$ is the generator of the SSEP. $L_\alpha$ and $L_\beta$ are the generators of the Glauber dynamics which act at sites 1 and $n-1$. Their actions on functions $f : \Omega_n \to \mathbb{R}$ are

$$\begin{aligned}
L_S f(\eta) &= \sum_{x=1}^{n-2} \{a_{x,x+1}(\eta) + a_{x+1,x}(\eta)\} (f(\eta^{x,x+1}) - f(\eta)), \\
L_\alpha f(\eta) &= \frac{m}{n^\theta} \{\alpha(1-\eta(1)) + (1-\alpha)\eta(1)\} (f(\eta^1) - f(\eta)), \\
L_\beta f(\eta) &= \frac{m}{n^\theta} \{\beta(1-\eta(n-1)) + (1-\beta)\eta(n-1)\} (f(\eta^{n-1}) - f(\eta)),
\end{aligned} \tag{6}$$

where the flip configuration $\eta^x$ is given by

$$\eta^x(z) = \begin{cases} \eta(z), \ z \neq x, \\ 1-\eta(x), \ z = x. \end{cases}$$

*Remark 1.* Recall that the diffusion coefficient of the PME is given by $D(\rho) = M\rho^{M-1}$, for $M > 1$. We note that the exchange rate in (3) is related to the diffusion coefficient of the PME when $M = 2$. Considering different values of $M$ (including $M = 1$), we have to consider different exchange rates, for example:

| $M$ | $D(\rho)$ | $c_{x,x+1}(\eta)$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | $2\rho$ | $\eta(x-1) + \eta(x+2)$ |
| 3 | $3\rho^2$ | $\eta(x-2)\eta(x-1) + \eta(x-1)\eta(x+2) + \eta(x+2)\eta(x+3)$ |

*Remark 2.* Throughout the text we denote by $\{\eta_{tn^2}\}_{t\geq 0}$ the Markov process sped up in the diffusive time scale $tn^2$.

*Remark 3.* The results presented here are also valid for any integer number $M > 2$. For a deeper discussion about the model we refer the reader to [3].

*Remark 4.* The dynamics of the PMM is degenerate (due to the constraints of the jump rates) and do not conserve the total number of particles (due to the Glauber dynamics). However, since the PMM is superposed with a SSEP dynamics, it becomes an irreducible Markov process and therefore only one invariant measure exists. In the equilibrium state ($\alpha = \beta$), it is not difficult to see that the Bernoulli product measure with a constant parameter ($\rho = \alpha = \beta$) is a reversible measure, and in particular, it is invariant. But in the non-equilibrium state ($\alpha \neq \beta$), we have no information about the invariant measure of the process. We stress that we have been trying different approaches in order to have some information about it but without success. One of them was to use the matrix ansatz, introduced in [8], which we could not apply due to the complicated action of the bulk dynamics. Another one is to use duality theory for IPS [11], we start the work here via a description of the PMM which uses algebra representation theory; nevertheless this is still a work in progress.

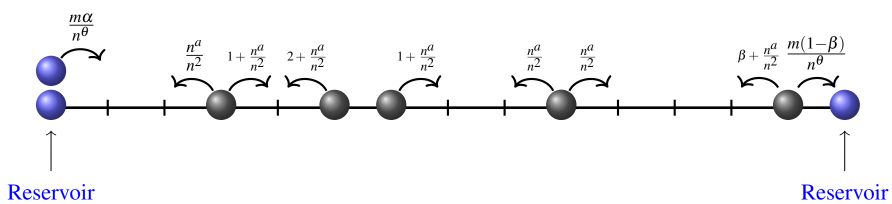


**Fig. 1** Allowed jumps for the perturbed porous medium model with slow reservoirs (with $M = 2$)

## 2 Fick's law for the PMM with slow reservoirs

Throughout this section we will work with the open PMM, introduced in the previous subsection. Moreover, since we are assuming the validity of the hydrodynamic limit proved in [3] using the Entropy method of [14], we need to avoid blocked configurations in order to have an irreducible Markov process. For this reason we superposed the PMM dynamics with a SSEP dynamics with a time scale slower than the diffusive one. This guarantees that, when scaling the time diffusively, we can see the impact of the SSEP at the microscopic level, but we cannot see it at the macroscopic level.

### 2.1 Currents

Let $\eta \in \Omega_n$. We denote by $j_{x,x+1}(\eta)$ the instantaneous current of particles over the bond $\{x, x+1\}$. In other words, it is the rate at which the particle jumps from the site $x$ to $x+1$, minus the rate at which the particle jumps from the site $x+1$ to $x$. Thus, for $x \in \Sigma_{n-1}$, the current in the bulk is given by

$$j_{x,x+1}(\eta) = (\eta(x) - \eta(x+1))(\eta(x-1) + \eta(x+2) + n^{a-2}). \tag{7}$$

In the same manner, the current over the bond in the left (resp. right) boundary is given by

$$j_{0,1}(\eta) = \tfrac{m}{n^\theta}(\alpha - \eta(1)) \quad \text{and} \quad j_{n-1,n}(\eta) = \tfrac{m}{n^\theta}(\eta(n-1) - \beta).$$

Now, we look for a local function $h : \Omega_n \to \mathbb{R}$, such that for every $x \in \Sigma_{n-1}$ the current can be written as $j_{x,x+1}(\eta) = \tau_x h(\eta) - \tau_{x+1} h(\eta)$, where $\tau_x h(\eta) = h(\tau_x \eta)$. The function $\tau_x$ being the translation by $x$ in the configuration $\eta$. If such a function exists, the Dynkin martingale will be much easier to compute (see (23)), since we can sum by parts and transfer the gradient to the test function. Models for which the current is the gradient of a local function are called gradient models, see for instance [18]. Hence, summing and subtracting $\eta(x)\eta(x+1)$ in (7), we can write it as

$$j_{x,x+1}(\eta) = \tau_x h(\eta) - \tau_{x+1} h(\eta), \tag{8}$$

where

$$\tau_x h(\eta) = \eta(x-1)\eta(x) + \eta(x)\eta(x+1) - \eta(x-1)\eta(x+1) + n^{a-2}\eta(x). \tag{9}$$

Therefore, the PMM is a gradient model.

### 2.2 Integrated currents

Let $t \in [0,T]$, for $T > 0$. For any $x \in \Sigma_n \cup \{0\}$, we denote by $N_t^n(x)$ the total number of particles that jumped from site $x$ to $x+1$ in an interval of time $[0, tn^2]$, and by $\tilde{N}_t^n(x)$ the total number of particles that jumped from site $x+1$ to $x$ in the same time interval. Thus, we define the integrated current at time $t$ and location $x$ by

$$J_t^n(x) := N_t(x) - \tilde{N}_t(x), \quad \text{for } x \in \Sigma_n \cup \{0\}. \tag{10}$$

In other words, $J_t^n(x)$ denotes the flux of particles through the bond $\{x, x+1\}$ in an interval of time $[0, tn^2]$. The integrated current (10) can be written in terms of its conservative and non-conservative parts. We denote by $Q_t^n(x)$ the conservative integrated current at time $t$ and location $x$, which records the particle jumps from the diffusive part of the dynamics (PMM and SSEP)

$$Q_t^n(x) := J_t^n(x), \quad \text{for } x \in \Sigma_{n-1}. \tag{11}$$

We denote by $K_t^n(x)$ the non-conservative integrated current at time $t$ and location $x$, which records the particles inserted and removed from the system at sites 1 or $n-1$ (Glauber dynamics)

$$K_t^n(x) := J_t^n(x), \quad \text{for } x = 0, n-1. \tag{12}$$

Having disposed of this preliminary step, we can now define the infinitesimal generator of the joint process $\{\eta_t, J_t^n(x)\}_{t\geq 0}$ as

$$\begin{aligned}\tilde{L}_n f(\eta, J^n(x)) = \tilde{L}_P f(\eta, J^n(x)) + n^{a-2}\tilde{L}_S f(\eta, J^n(x)) \\ + \tilde{L}_\alpha f(\eta, J^n(x)) + \tilde{L}_\beta f(\eta, J^n(x)),\end{aligned} \tag{13}$$

for $x \in \Sigma_n \cup \{0\}$. To simplify the notation, let $p_{x,x+1}(\eta) = a_{x,x+1}(\eta)(c_{x,x+1}(\eta) + n^{a-2})$. For each $x \in \Sigma_{n-1}$, we define the part of (13) corresponding to the jumps in the bulk as

$$\begin{aligned}\left(\tilde{L}_P + n^{a-2}\tilde{L}_S\right) f(\eta, J^n(x)) &= p_{x,x+1}(\eta)\left(f(\eta^{x,x+1}, J^n(x)+1) - f(\eta, J^n(x))\right) \\ &+ p_{x+1,x}(\eta)\left(f(\eta^{x,x+1}, J^n(x)-1) - f(\eta, J^n(x))\right) \\ &+ \sum_{\substack{y\in\Sigma_{n-1} \\ y\neq x}} \left(p_{y,y+1}(\eta) + p_{y+1,y}(\eta)\right)\left(f(\eta^{y,y+1}, J^n(y)) - f(\eta, J^n(y))\right),\end{aligned} \tag{14}$$

and the part of (13) corresponding to the jumps in the boundaries as

$$\begin{aligned}\tilde{L}_\alpha f(\eta, J^n(0)) &= \frac{m}{n^\theta}\Big(\alpha(1-\eta(1))\left(f(\eta^1, J^n(0)+1) - f(\eta, J^n(0))\right)\Big) \\ &+ \frac{m}{n^\theta}\Big((1-\alpha)\eta(1)\left(f(\eta^1, J^n(0)-1) - f(\eta, J^n(0))\right)\Big),\end{aligned}$$

$$\begin{aligned}\tilde{L}_\beta f(\eta, J^n(n-1)) &= \frac{m}{n^\theta}\Big(\beta(1-\eta(n-1))\left(f(\eta^{n-1}, J^n(n-1)-1) - f(\eta, J^n(n-1))\right) \\ &+ (1-\beta)\eta(n-1)\left(f(\eta^{n-1}, J^n(n-1)+1) - f(\eta, J^n(n-1))\right)\Big).\end{aligned} \tag{15}$$

*Remark 5.* Throughout this section the process is sped up in the diffusive time scale $tn^2$.

*Remark 6.* If we take $f(\eta, J) = f(\eta)$ in (14) and (15), we recover the infinitesimal generator of $\{\eta_t\}_{t\geq 0}$, which is defined in (5). Moreover, if we take $f$ being the projection in the second variable, that is, $f(\eta, J) = J$ in (14) and (15), we recover the instantaneous current through the bond $\{x, x+1\}$ as we can see below.

For $x \in \Sigma_{n-1}$, we have

$$\begin{aligned}\left(\tilde{L}_P + n^{a-2}\tilde{L}_S\right) J^n(x) &= a_{x,x+1}(\eta)(c_{x,x+1}(\eta) + n^{a-2})\,(J^n(x) + 1 - J^n(x)) \\ &\quad + a_{x+1,x}(\eta)(c_{x+1,x}(\eta) + n^{a-2})\,(J^n(x) - 1 - J^n(x)) \\ &= j_{x,x+1}(\eta).\end{aligned}$$

For the left (resp. right) boundary, we have

$$\begin{aligned}\tilde{L}_\alpha J^n(0) &= \frac{m}{n^\theta}\Big(\alpha(1-\eta(1))\,(J^n(0)+1-J^n(0))\Big) \\ &\quad + \frac{m}{n^\theta}\Big(((1-\alpha)\eta(1))\,(J^n(0)-1-J^n(0))\Big) \\ &= j_{0,1}(\eta),\end{aligned}$$

$$\begin{aligned}\tilde{L}_\beta J^n(n-1) &= \frac{m}{n^\theta}\Big(\beta(1-\eta(n-1))\,(J^n(n-1)-1-J^n(n-1))\Big) \\ &\quad + \frac{m}{n^\theta}\Big(((1-\beta)\eta(n-1))\,(J^n(n-1)+1-J^n(n-1))\Big) \\ &= j_{n-1,n}(\eta).\end{aligned}$$

## *2.3 Empirical measures*

Fix $t \in [0,T]$. For $\eta \in \Omega_n$, we define the empirical measure $\pi_t^n$ on $[0,1]$ as

$$\pi_t^n := \frac{1}{n}\sum_{x\in\Sigma_n} \eta_t(x)\delta_{x/n}, \tag{16}$$

where $\delta_u$ is the Dirac measure concentrated on $u \in [0,1]$. Recall the definition of the conservative current (11). The empirical measure associated with this current is defined as the signed measure on $[0,1]$

$$Q_t^n := \frac{1}{n^2}\sum_{x=1}^{n-2} Q_t^n(x)\delta_{x/n}. \tag{17}$$

Note that the renormalization factor of order $n^2$ arises in (17) because we need to take into account the space renormalization and the diffusive scaling of the PMM and SSEP dynamics. Now, recall the definition of (12). The empirical measure associated with this boundary current is defined as

$$K_t^n := \frac{1}{n}K_t^n(0)\delta_{0/n} + \frac{1}{n}K_t^n(n-1)\delta_{n-1/n}. \tag{18}$$

Since expression (18) is related to the Glauber part of the process, we only need to take into account the space renormalization factor of order $n$.
Let $f \in C^1([0,1])$ be a test function. We define the empirical density of particles at time $t$, that is, the integral of $f$ with respect to the empirical measure $\pi_t^n$, as

$$\pi_t^n(f) = \frac{1}{n}\sum_{x\in\Sigma_n} f\left(\tfrac{x}{n}\right)\eta_t(x).$$

In the same manner, the current field $J_t^n$ is defined as

$$J_t^n(f) := Q_t^n(f) + K_t^n(f),$$

where $Q_t^n$ is the conservative current field

$$Q_t^n(f) := \frac{1}{n^2}\sum_{x=1}^{n-2} f\left(\tfrac{x}{n}\right) Q_t^n(x),$$

and $K_t^n$ is the non-conservative current field

$$K_t^n(f) := \frac{1}{n}\left(f(0)K_t^n(0) - f\left(\tfrac{n-1}{n}\right)K_t^n(n-1)\right).$$

## *2.4 Fick's law*

In this section, we define the notion of weak solution of the PME with Dirichlet, Robin, and Neumann boundary conditions, and we state the Fick's law for the PMM with slow reservoirs. Before we start, let us fix some notations. We denote by:

- $C_c^\infty(0,1)$, the set of all real-valued functions $G \in C^\infty(0,1)$ with compact support;
- $C_0^{1,2}([0,T]\times[0,1])$, the set of all real-valued functions $G \in C^{1,2}([0,T]\times[0,1])$ such that $G_s(0) = G_s(1) = 0$, for all $s \in [0,T]$;
- $\langle\cdot,\cdot\rangle$, the inner product in $L^2([0,1])$ with corresponding norm $\|\cdot\|_2$.

**Definition 1.** Let $\mathscr{H}^1$ be the set of all locally summable functions $\zeta : [0,1] \to \mathbb{R}$ such that there exists a function $\partial_u\zeta \in L^2([0,1])$ satisfying

$$\langle \partial_u G, \zeta\rangle = \langle G, \partial_u\zeta\rangle,$$

for all $G \in C_c^\infty(0,1)$. For $\zeta \in \mathscr{H}^1$, we define the norm

$$\|\zeta\|_{\mathscr{H}^1} := \left(\|\zeta\|_2^2 + \|\partial_u\zeta\|_2^2\right)^{1/2}.$$

Let $L^2(0,T;\mathscr{H}^1)$ be the set of all measurable functions $\xi : [0,T] \to \mathscr{H}^1$ such that

$$\|\xi\|^2_{L^2(0,T;\mathscr{H}^1)} := \int_0^T \|\xi_t\|^2_{\mathscr{H}^1}\,dt < \infty.$$

**Definition 2.** Let $T > 0$, $\alpha,\beta \in (0,1)$ and $g : [0,1] \to [0,1]$ a measurable function. We say that $\rho : [0,T]\times[0,1] \to [0,1]$ is a weak solution of the PME with Dirichlet boundary conditions

$$\begin{cases} \partial_t \rho_t(u) = \Delta\,(\rho_t(u))^2, \quad (t,u) \in (0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, \quad t \in (0,T], \\ \rho_0(u) = g(u), \quad u \in [0,1], \end{cases} \tag{19}$$

if the following conditions hold:

1. $\rho^2 \in L^2(0,T;\mathscr{H}^1)$;
2. $\rho$ satisfies the integral equation:

$$\int_0^1 \left\{\rho_t(u)G_t(u) - g(u)G_0(u)\right\} du - \int_0^t \int_0^1 \left\{\rho_s(u)\partial_s G_s(u) + (\rho_s)^2(u)\Delta G_s(u)\right\} du\,ds \\ + \int_0^t \left\{\beta^2 \partial_u G_s(1) - \alpha^2 \partial_u G_s(0)\right\} ds = 0,$$

   for all $t \in [0,T]$ and any function $G \in C_0^{1,2}([0,T]\times[0,1])$;
3. $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$ for all $t \in (0,T]$.

**Definition 3.** Let $T > 0$, $\kappa \geq 0$, $\alpha, \beta \in (0,1)$ and $g : [0,1] \to [0,1]$ a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the PME with Robin boundary conditions

$$\begin{cases} \partial_t \rho_t(u) = \Delta\,(\rho_t(u))^2, \quad (t,u) \in (0,T] \times (0,1), \\ \partial_u(\rho_t(0))^2 = \kappa(\rho_t(0) - \alpha), \quad t \in (0,T], \\ \partial_u(\rho_t(1))^2 = \kappa(\beta - \rho_t(1)), \quad t \in (0,T], \\ \rho_0(u) = g(u), \quad u \in [0,1], \end{cases} \tag{20}$$

if the following conditions hold:

1. $\rho^2 \in L^2(0,T;\mathscr{H}^1)$;
2. $\rho$ satisfies the integral equation:

$$\int_0^1 \left\{\rho_t(u)G_t(u) - g(u)G_0(u)\right\} du - \int_0^t \int_0^1 \left\{\rho_s(u)\partial_s G_s(u) + \rho_s^2(u)\Delta G_s(u)\right\} du\,ds \\ + \int_0^t \left\{(\rho_s(1))^2 \partial_u G_s(1) - (\rho_s(0))^2 \partial_u G_s(0)\right\} ds \\ - \kappa \int_0^t \left\{G_s(0)(\alpha - \rho_s(0)) + G_s(1)(\beta - \rho_s(1))\right\} ds = 0,$$

   for all $t \in [0,T]$ and any function $G \in C^{1,2}([0,T]\times[0,1])$.

Before stating the Fick's law let us fix some notations. Let $\mathscr{M}_+$ be the space of positive measures on $[0,1]$ with total mass bounded by 1 equipped with the weak topology. Let $\mu_n$ be measure on $\Omega_n$. We denote by $\mathbb{P}_{\mu_n}$ the probability measure in the Skorokhod space $\mathscr{D}([0,T],\Omega_n)$, induced by the accelerated Markov process $\{\eta_{tn^2}\}_{t\geq 0}$ and the initial measure $\mu_n$. We denote by $\mathbb{E}_{\mu_n}$ the expectation with respect to $\mathbb{P}_{\mu_n}$.

Let $g:[0,1]\to[0,1]$ be a measurable function. For each $n\in\mathbb{N}$, we say that $\{\mu_n\}_{n\in\mathbb{N}}$ is associated with $g(\cdot)$, if for any continuous function $H:[0,1]\to\mathbb{R}$ and any $\delta>0$:

$$\lim_{n\to+\infty}\mu_n\left(\eta\in\Omega_n:\left|\frac{1}{n}\sum_{x\in\Sigma_n}H(\tfrac{x}{n})\eta(x)-\int_0^1 H(u)g(u)\,du\right|>\delta\right)=0. \tag{21}$$

**Theorem 1 (Fick's law).** *Fix $\theta\in[0,+\infty)$. Let $g:[0,1]\to[0,1]$ be a measurable function, $H:[0,1]\to\mathbb{R}$ a continuous function, and $\{\mu_n\}_{n\in\mathbb{N}}$ a sequence of probability measures on $\Omega_n$ associated with $g(\cdot)$, as in* (21)*. Then, for any $t\in[0,T]$ and any $\delta>0$, we have*

$$\lim_{n\to+\infty}\mathbb{P}_{\mu_n}\Big(\eta_\cdot\in\mathscr{D}([0,T],\Omega_n):\Big|\frac{1}{n^2}\sum_{x=1}^{n-2}Q_t^n(x)H\left(\tfrac{x}{n}\right)-\int_0^1 H(u)\nabla\rho_t^2(u)\,du\Big|>\delta\Big)=0,$$

$$\begin{aligned}\lim_{n\to+\infty}\mathbb{P}_{\mu_n}\Big(\eta_\cdot\in\mathscr{D}([0,T],\Omega_n):\Big|&\frac{1}{n}\left(H(0)K_t^n(0)-H\left(\tfrac{n-1}{n}\right)K_t^n(n-1)\right)\\&-\mathbb{1}_{\{\theta=1\}}\kappa\int_0^t H(0)(\alpha-\rho_s(0))+H(1)(\beta-\rho_s(1))ds\Big|>\delta\Big)=0,\end{aligned}$$

*where*

- $\rho_t(u)$ *is a weak solution of* (19)*, for* $0\le\theta<1$*;*
- $\rho_t(u)$ *is a weak solution of* (20) *($\kappa=m$), for* $\theta=1$*;*
- $\rho_t(u)$ *is a weak solution of* (20) *(with $\kappa=0$), for* $\theta>1$.

*Remark 7.* Note that $J_t^n=Q_t^n+K_t^n$. From the previous theorem we have that $J^n$ converges weakly to $J\,du$, where $J$ is the weak solution of

$$J=-D(\rho)\nabla\rho=-\nabla\rho^2.$$

The result stated in Theorem 1, that we will prove in the next section, is the Law of large numbers for the empirical measures defined in (17) and (18). It is the analog of the Law of large numbers for the empirical measure (16), known in the literature as hydrodynamic limit.

## 2.5 Proof of Theorem 1

In this section, we prove Theorem 1, that is, the validity of Fick's law: the currents which enter and exit from the system are at all times equal to the local density gradient at 0 and 1. In order to prove it we need to assume the validity of the hydrodynamic limit and some technical results, known as replacement lemmas, which are stated in the appendix. These results and the hydrodynamic limit are proved in [3]. The theorems we refer to in the proof are stated in the first and second sections of the Appendix.

*Proof.* Let us prove the first identity of the theorem. Our proof starts with the observation that by Dynkin's formula, see Lemma A1.5.1 of [18], for a fixed test function $H \in C^1([0,1])$, we have that

$$M_t^n(H) = Q_t^n(H) - Q_0^n(H) - \int_0^t n^2 \tilde{L}_n Q_s^n(H)\, ds, \tag{22}$$

is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$, which vanishes as $n \to \infty$ in $L^2(\mathbb{P}_{\mu_n})$ (see fist section of the Appendix). Note that $Q_0^n(H) = 0$. Hence, we can write (22) as

$$Q_t^n(H) - \int_0^t \sum_{x=1}^{n-2} H\left(\tfrac{x}{n}\right) j_{x,x+1}(\eta_{sn^2})\, ds.$$

Since the PMM is a gradient model, performing a summation by parts in the previous expression, we can write (22) as

$$Q_t^n(H) - \int_0^t \frac{1}{n} \sum_{x=1}^{n-2} \nabla_n^- H\left(\tfrac{x}{n}\right) \tau_x h(\eta_{sn^2}) + H\left(\tfrac{0}{n}\right) \tau_1 h(\eta_{sn^2}) - H\left(\tfrac{n-1}{n}\right) \tau_{n-1} h(\eta_{sn^2})\, ds, \tag{23}$$

where $\tau_x h(\eta_{sn^2})$ is defined in (9) and

$$\nabla_n^+ H\left(\tfrac{x}{n}\right) = n\left(H\left(\tfrac{x+1}{n}\right) - H\left(\tfrac{x}{n}\right)\right), \quad \nabla_n^- H\left(\tfrac{x}{n}\right) = n\left(H\left(\tfrac{x}{n}\right) - H\left(\tfrac{x-1}{n}\right)\right). \tag{24}$$

Thus, we want to examine the convergence of (23) for each value of $\theta \in [0, +\infty)$.
If $\theta < 1$, the test function vanishes at the boundary. From the hydrodynamic limit and Theorem 3, we have that the integral term of (23) converges in $\mathbb{P}_{\mu_n}$, as $n \to \infty$ to

$$\begin{aligned}\int_0^t \int_0^1 \nabla H(u) \rho_s^2(u)\, du\, ds = &\int_0^t H(1)\rho_s^2(1) - H(0)\rho_s^2(0)\, ds \\ &- \int_0^t \int_0^1 H(u) \nabla \rho_s^2(u)\, du\, ds,\end{aligned}$$

which is equal to $-\int_0^t \int_0^1 H(u)\nabla\rho_s^2(u)\, du\, ds$.
If $\theta \geq 1$, the test function does not necessarily vanishes at the boundary. From the hydrodynamic limit, Theorems 3 and 4 it follows that the integral term of (23) converges in $\mathbb{P}_{\mu_n}$, as $n \to \infty$ to

$$\int_0^t \int_0^1 \nabla H(u) \rho_s^2(u)\, du\, ds + \int_0^t H(0)\rho_s^2(0) - H(1)\rho_s^2(1)\, ds,$$

which is also equal to $-\int_0^t \int_0^1 H(u)\nabla\rho_s^2(u)\, du\, ds$.
In the same manner, for $H \in C^1([0,1])$ we have that

$$\tilde{M}_t^n(H) = K_t^n(H) + \kappa \int_0^t \frac{n}{n^\theta} \left(H(\tfrac{1}{n})(\alpha - \eta_{sn^2}(1)) + H(\tfrac{n-1}{n})(\beta - \eta_{sn^2}(n-1))\right) ds, \tag{25}$$

is also a martingale that vanishes in $L^2(\mathbb{P}_{\mu_n})$ as $n \to \infty$, see first section of the Appendix. Let us now examine the convergence of the integral term of (25), for each value of $\theta \in [0,+\infty)$.
If $0 < \theta < 1$, the test function vanishes at the boundary, and by a Taylor expansion on $H$ we get

$$\frac{\kappa}{n^\theta}\int_0^t -\nabla_n^+ H(0)(\alpha - \eta_{sn^2}(1)) - \nabla_n^- H(1)(\beta - \eta_{sn^2}(n-1))\,ds,$$

where $\nabla_n^{\pm} H\left(\frac{x}{n}\right)$ are defined in (24). The previous expression is bounded from above by

$$\frac{\kappa}{n^\theta}\|\nabla H\|_\infty \int_0^t |\alpha - \eta_{sn^2}(1)| + |\beta - \eta_{sn^2}(n-1)|\,ds,$$

which vanishes as $n \to \infty$. If $\theta = 0$, the test function vanishes at the boundary, and by Theorem 5 we have that the integral term of (25) vanishes. If $\theta = 1$, the test function does not vanishes at the boundary, and from Theorem 6 we have that the integral term of (25) converges in $\mathbb{P}_{\mu_n}$, as $n \to \infty$ to

$$\kappa\int_0^t H(0)(\alpha - \rho(s,0)) + H(1)(\beta - \rho(s,1))\,ds.$$

Finally, if $\theta > 1$, we have that the integral term of (25) is bounded from above by

$$\frac{\kappa}{n^{\theta-1}}\|H\|_\infty \int_0^t |\alpha - \eta_{sn^2}(1)| + |\beta - \eta_{sn^2}(n-1)|\,ds,$$

which vanishes as $n \to \infty$, concluding the proof.

## 3 Stochastic duality relations for the PMM

In order to show our self-duality result for the PMM, we first need to give some context regarding stochastic duality theory for Markov jumping processes. The idea behind duality is to get information on a given process from another process, its *dual*. The link between these two processes is provided by a set of so-called *duality functions*, i.e. a set of observables that are functions of both processes and whose expectations satisfy the following definition.

**Definition 4 (Duality of processes).** For $t \geq 0$, let $\eta_t$ and $\xi_t$ be two continuous time Markov processes with state spaces $\Omega$ and $\Omega^{dual}$ , respectively. We say that $\xi_t$ is *dual* to $\eta_t$ with duality function $D : \Omega \times \Omega^{dual} \to \mathbb{R}$ if

$$\mathbb{E}_\eta[D(\eta_t,\xi)] = \mathbb{E}_\xi[D(\eta,\xi_t)]\ , \tag{26}$$

for all $(\eta, \xi) \in \Omega \times \Omega^{dual}$ and $t \geq 0$. In (26) $\mathbb{E}_\eta$ (respectively $\mathbb{E}_\xi$) is the expectation with respect to the law of the $\eta_t$ process initialized at $\eta$ (respectively the $\xi_t$ process initialized at $\xi$).

If $\eta_t$ and $\xi_t$ are two independent copies of the same process, we say that $\eta_t$ is *self-dual* with self-duality function $D$. We will see that this is the case for the bulk dynamics of the PMM. Indeed, self-duality can always be thought as a special case of duality where the dual process is an independent copy of the first one. The simplification of self-duality typically arises from the fact that in the copy process only a small number of particles are considered. Given the one-to-one correspondence between Markov processes and their semigroups, then one sees that a duality relation between two Markov processes is equivalent to a duality relation between their Markov semigroups, i.e.

$$(T_t D(\cdot, \xi))(\eta) = \left(T_t^{dual} D(\eta, \cdot)\right)(\xi) \ , \text{ for } t \geq 0, \tag{27}$$

where $T_t$ denotes the semigroup of the original process $\eta$ and $T_t^{dual}$ the semigroup of the dual process $\xi$. In the context of IPS duality can be defined at the level of their Markov generator, this is usually a definition easier to work with and the equivalence of these two definitions has been proved in [16].

**Definition 5 (Duality of generators).** For $t \geq 0$, let $L$ and $L^{dual}$ be generators of the two Markov processes $\eta_t$ and $\xi_t$, respectively. We say that $L^{dual}$ is *dual* to $L$ with duality function $D : \Omega \times \Omega^{dual} \longrightarrow \mathbb{R}$ if

$$[LD(\cdot, \xi)](\eta) = [L^{dual} D(\eta, \cdot)](\xi) \tag{28}$$

where we assume that both sides are well defined.

In case $L = L^{dual}$ we shall say that the process is *self-dual* and the self-duality relation becomes

$$[LD(\cdot, \xi)](\eta) = [LD(\eta, \cdot)](\xi) \ . \tag{29}$$

In equation (28) (respectively (29)) it is understood that $L$ on the left hand side acts on $D$ as a function of the first variable $\eta$, while $L^{dual}$ (resp. $L$) on the right hand side acts on $D$ as a function of the second variable $\xi$. Definition 5 is easier to prove, so we will usually work under the assumption that the notion of duality (respectively self-duality) is the one in equation (28) (respectively (29)).
If the original process $\eta_t$ and the dual process $\xi_t$ are Markov processes with countable state space $\Omega$ and $\Omega^{dual}$ respectively, then the duality relation is equivalent to

$$\sum_{\eta' \in \Omega} L(\eta, \eta') D(\eta', \xi) = \sum_{\xi' \in \Omega^{dual}} (L^{dual})^T (\xi', \xi) D(\eta, \xi') \ , \tag{30}$$

where $L^T$ denotes the transposition of the generator $L$. In matrix notation (30) becomes

$$LD = D(L^{dual})^T \ . \tag{31}$$

Once more, if $L^{dual} = L$ we obtain the corresponding definition for self-duality. In this context, the generator $L$ is given by a matrix known as *rate matrix* such that

$$L(\eta,\eta') \geq 0 \qquad \text{and} \qquad \sum_{\eta'} L(\eta,\eta') = 0\,.$$

For $\eta \neq \eta'$, we say that the process jumps from $\eta$ to $\eta'$ with *rate* $L(\eta,\eta')$.

*Remark 8.* Given that the PMM has a finite state space, the self-duality relations, which will be characterized in the following two sections, read as in equation (31).

Our goal is to frame and find a self-duality relation for the PMM. This is achieved via an algebraic approach, first proposed in [11] and further developed in [4, 10, 13], which relies on the following idea. It starts from the hypothesis that the Markov generator is an element of the universal enveloping algebra of a Lie algebra. Then the derivation of a (self-)duality relation is based on two structural ideas:

i) duality can be seen as a change of representation of a Lie algebra: more precisely one moves between two equivalent representations and the *intertwiner* of those representations yields the duality function.
ii) self-duality is related to the reversibility of the process and the existence of an algebra element that commutes with the generator of the process.

We will make use of item ii) to find a self-duality function for the bulk generator of the PMM in equation (2).

In what follows, classical theorems and propositions are taken from [10, 11].

## *3.1 Algebraic approach to duality*

In this section, we briefly recall the idea to find self-duality relation for an IPS with a reversible measure, see [11]. For the PMM the existence of a reversible measure can be found by the detailed balance equations and it is the starting point of our analysis. As stated in Remark 4, the reversible measure is known for the open system only in case the reservoirs are tuned with the same parameter. In the case of a closed system, the reversible measure has the same form, namely the product of Bernoulli distribution, with a free constant parameter.

### 3.1.1 Symmetries and self-duality

In this subsection, we review the general techniques to exhibit a self-duality relation for an IPS.

**Definition 6.** Let $A$ and $B$ be two matrices having the same dimension. We say that $A$ is a symmetry of $B$ if $A$ commutes with $B$, i.e.

$$[A,B] := AB - BA = 0 \, .$$

Clearly the identity matrix is always a symmetry and it is easy to verify that $[AB,CD] = A[B,C]D + CA[B,D] + [A,C]BD + C[A,D]B$. The main idea is that self-duality (in the context of Markov processes with countable state space) can be recovered starting from a *trivial duality* which is based on the reversible measures of the corresponding process. One then can act with a symmetry of the model on this trivial self-duality and turn it into a non-trivial one. The following theorem formalizes this last idea.

**Proposition 1 (Symmetries and self-duality).** *Let $d$ be a self-duality function of the generator $L$ and let $S$ be a symmetry of $L$, then $D = Sd$ is again a self-duality function for $L$.*

*Proof.* The proof follows from a straightforward computation and in matrix notation it reads

$$LD = LSd = SLd = SdL^T = DL^T \, ,$$

where the second identity follows from the fact that $S$ and $L$ commutes, while the third one is due to the self-duality of the generator $L$ with self-duality function $d$.

If there is a description of the process generator in terms of a Lie algebra, then symmetries can be constructed using this algebraic structure. Notice that the two main elements of the theorem above are the initial self-duality $d$ and the symmetry operator $S$. We explain now how these two objects can easily be found whenever reversibility and an algebraic description of the process are available. In general, if the process has a reversible measure, the self-duality $d$ can be easily found starting from the reversibility, as the following proposition shows.

**Proposition 2.** *If the process associated with the generator $L$ has reversible measure $\mu$, then the function $d : \Omega \times \Omega \to \mathbb{R}$*

$$d(\eta,\xi) = \frac{\delta_{\eta,\xi}}{\mu(\eta)} \tag{32}$$

*is a self-duality function.*

*Proof.* The proof follows from the reversibility of the measure $\mu$. Since we are on a countable state space, we can use the notion of self-duality via the matrix notation in equation (31). Namely,

$$Ld = dL^T$$

which reads

$$\sum_{\eta' \in \Omega} L(\eta,\eta')d(\eta',\xi) = \sum_{\xi' \in \Omega} d(\eta,\xi')L^T(\xi',\xi) \, ,$$

once we substitute the expression of $d$ as in equation (32) we get

$$\sum_{\eta' \in \Omega} L(\eta,\eta')\frac{\delta_{\eta',\xi}}{\mu(\xi)} = \sum_{\xi' \in \Omega} \frac{\delta_{\eta,\xi'}}{\mu(\eta)} L^T(\xi',\xi) \, .$$

The sum on the left-hand side only survives for $\eta' = \xi$ while the one on the right-hand side only survives for $\xi' = \eta$, i.e,

$$L(\eta,\xi)\frac{1}{\mu(\xi)} = L(\xi,\eta)\frac{1}{\mu(\eta)},$$

which is exactly the detailed balance condition.

We refer to this diagonal self-duality function in equation (32) as trivial or cheap self-duality function. At this point one may now wonder how the operator $S$ is found; here it is where the properties of the algebra help. If the algebra admits the Casimir element $\mathscr{C}$ then it is not hard to find symmetries for the process generator. Indeed, since the Casimir element commutes with all the other elements of the algebra, then any element of the algebra is potentially a good candidate as a symmetry of $\mathscr{C}$. Moreover, whenever the process generator $L$ can be written as the coproduct of the Casimir element, then a symmetry of the Casimir can be extended using its coproduct to a symmetry of the generator as shown in the proposition below. It will also be useful to recall that the coproduct of a Lie algebra generator $X$, denoted by $\Delta(X)$ is defined via the tensor product $\otimes$, as

$$\Delta(X) = 1 \otimes X + X \otimes 1 \tag{33}$$

and that it can be extended as an algebra homomorphism to the universal enveloping algebra.

**Proposition 3.** *If $S$ is a symmetry of the central element $\mathscr{C}$, then $\Delta(S)$ is a symmetry for $\Delta(\mathscr{C})$.*

*Proof.* Starting from $[\mathscr{C}, S] = 0$, we want to show that $[\Delta(\mathscr{C}), \Delta(S)] = 0$. This follows from the fact that the coproduct is an algebra homomorphism, i.e.

$$[\Delta(\mathscr{C}), \Delta(S)] = \Delta(\mathscr{C})\Delta(S) - \Delta(S)\Delta(\mathscr{C}) = \Delta(\mathscr{C}S) - \Delta(S\mathscr{C}) = \Delta(\mathscr{C}S - S\mathscr{C}) = 0\,.$$

### 3.1.2 The Lie algebra $\mathfrak{su}(2)$

In this subsection, we link our process to the Lie algebra $\mathfrak{su}(2)$, for which the Casimir element exists. The $\mathfrak{su}(2)$ Lie algebra is generated by three abstract operators, namely $J^0$, $J^+$ and $J^-$, which satisfy the following commutation relations

$$[J^0, J^{\pm}] = \pm J^{\pm} \qquad \text{and} \qquad [J^+, J^-] = 2J^0\,, \tag{34}$$

while the adjoint are given by $(J^0)^* = J^0$, $(J^+)^* = J^-$ and $(J^-)^* = J^+$. The Casimir element is

$$\mathscr{C} = 2(J^0)^2 + J^+J^- + J^-J^+\,. \tag{35}$$

It is easy to check that $\mathscr{C}$ is self-adjoint, i.e. $\mathscr{C} = \mathscr{C}^*$ and it commutes with any generators of the algebra. We propose here two different notations that satisfy the rules of the $\mathfrak{su}(2)$ algebra. The first one is defined by the action of the three matrices on vectors of the natural basis of $\mathbb{R}^2$, $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$. In bra-ket notation, it becomes

$$\begin{cases} J^+ \mid \eta_x\rangle = \ (1-\eta_x) \mid \eta_x+1\rangle, \\ J^- \mid \eta_x\rangle = \ \eta_x \mid \eta_x-1\rangle, \\ J^0 \ \mid \eta_x\rangle = \ (\eta_x-1/2) \mid \eta_x\rangle, \end{cases}$$

where the $\mid n\rangle$ here is a column vector that represents the $n^{th}$ element of the canonical basis of $\mathbb{R}^2$. Explicitly this means that we can think of the $\mathfrak{su}(2)$ generators as three $2\times 2$ matrices

$$J^+ = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \qquad J^- = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \qquad J^0 = \begin{pmatrix}-1/2 & 0\\ 0 & 1/2\end{pmatrix}.$$

The second representation, equivalent, acts of functions $f:\{0,1\}\to\mathbb{R}$ and is given by

$$\begin{cases} (J^+f)(\eta_x) = \eta_x f(\eta_x-1), \\ (J^-f)(\eta_x) = (1-\eta_x)f(\eta_x+1), \\ (J^0f)(\eta_x) = (\eta_x-1/2)f(\eta_x), \end{cases}$$

where $f(-1)=f(2)=0$. In the next section, we use these two representations to show how the PMM can be described using the algebra's generators $J^+$, $J^-$, and $J^0$. From the first representation the rate matrix will arise, while the second one is used to find the generators of section 1.2

### *3.2 Porous medium model described with the $\mathfrak{su}(2)$ algebra*

We now link together the previous two sections 3.1 and 3.2: we will show that it is possible to describe the PMM generator, $L_P$, using the three algebra generators $J^+$, $J^-$ and $J^0$. This is inspired by the algebraic description of the SSEP which we recall here. We start by considering two sites, labeled by 1 and 2, then the SSEP generator is

$$\begin{aligned} L_S f(\eta) = {} & [\eta_1(1-\eta_2)]\,[f(\eta_1-1,\eta_2+1)-f(\eta_1,\eta_2)] \\ & +(\eta_2(1-\eta_1)]\,[f(\eta_1+1,\eta_2-1)-f(\eta_1,\eta_2)]\ , \end{aligned} \tag{36}$$

for $\eta=(\eta_1,\eta_2)$. We will see that $L_S$ can be described via the two representations of the $\mathfrak{su}(2)$ algebra introduced above.

*Remark 9.* In principle $L_S$ is found by summing all over the lattice site $\Sigma_n$. However, with abuse of notation, but without loss of generality since the coproduct structure introduced in equation (33) can be generalized to any lattice size, we will refer to the SSEP generator for two sites only. Conversely, the minimum number of sites to describe the PMM is 4. At this stage, it is still not clear how to treat the duality relations for the Glauber dynamics, namely for the generators $L_\alpha$ and $L_\beta$ defined in equation (6). For this reason, it will be convenient to expand the space $\Sigma_n$ into the one dimensional discrete torus with $n$ points, $\mathbb{T}_n$.

The theorem below is saying that the SSEP bulk generator of equation (36) and the Casimir of the $\mathfrak{su}(2)$ of equation (35) are deeply related. An analog result holds for the PMM.

**Theorem 2.** *Given the structure of the $\mathfrak{su}(2)$ algebra described in the previous section, we can write the SSEP generator in terms of the coproduct of the Casimir element of the algebra in the following way*

$$L_S = \frac{1}{2}\Delta(\mathscr{C}) - \frac{1}{2}\otimes\mathscr{C} - \mathscr{C}\otimes\frac{1}{2} - \frac{1}{2}\ .$$

*Moreover, the term $-\frac{1}{2}\otimes\mathscr{C} - \mathscr{C}\otimes\frac{1}{2} - \frac{1}{2}$ on the right hand side of the previous display represents the identity times a constant.*

*Proof.* First we substitute the expression of the coproduct of the Casimir $\Delta(\mathscr{C})$, so that we get

$$L_S = J^+\otimes J^- + J^-\otimes J^+ + 2J^0\otimes J^0 - 1/2\ .$$

Using the first representation via matrices, we can write the rate matrix of the SSEP (for two sites) as

$$L_S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case, the expression $-\frac{1}{2}\otimes\mathscr{C} - \mathscr{C}\otimes\frac{1}{2} - \frac{1}{2}$ is the diagonal matrix with element $-2$. Equivalently, using the second notation, i.e. acting on the function of two variables we recover the well-known expression of equation (36). In this representation one can check that the action of the Casimir on function is $\mathscr{C}f(\eta) = \frac{3}{2}f(\eta)$, so that the expression $-\frac{1}{2}\otimes\mathscr{C} - \mathscr{C}\otimes\frac{1}{2} - \frac{1}{2}$ on functions of two variables just gives $-2f(\eta_1,\eta_2)$.

We now go back to the PMM. The first observation is that it works on 4 sites because even if the jumps only contemplate two central sites, there are two extra sites to be taken into account. For this reason, we start by considering the action on 4 sites only and so we restrict the analysis to the finite one-dimensional lattice $\Sigma_5$. Here we observe jumps between sites 2 and 3. In this setting the PMM generator acts on functions $f : \{0,1\}^{\Sigma_5} \to \mathbb{R}$. The key observation is that

$$\begin{aligned} L_P &= (J_1^0 + J_4^0 + I)L_S \\ &= \left(J_1^0 + J_4^0 + I\right)\left(J_2^+ J_3^- + J_2^- J_3^+ + 2J_2^0 J_3^0 - I/2\right)\ . \end{aligned} \tag{37}$$

Here the notation $J_x^a$ means that $J^a$ is acting on sites $x \in \Sigma_5$ for $a \in \{0,+,-\}$. Indeed, since the algebra generator $J^0$ does not increase or decrease the degree of the functions, we can use them to describe the extra constraint for the PMM. Here $L_S$ has to be thought as the SSEP generator acting on sites 2 and 3.
In the same spirit of Theorem 2 one can check that using the first representation we get the PMM rate matrix for 4 sites, namely

$$\begin{aligned} L_P = &\big(J^0 \otimes I \otimes I \otimes I + I \otimes I \otimes I \otimes J^0 + I\big) \\ &\cdot \left( I \otimes J^+ \otimes J^- \otimes I + I \otimes J^- \otimes J^+ \otimes I + 2 I \otimes J^0 \otimes J^0 \otimes I - \frac{1}{2} I \right) , \end{aligned} \tag{38}$$

here $I = I \otimes I \otimes I \otimes I$, is shorthand for the identity matrix of dimension 16. Using the second representation we get $L_P$, the PMM generator of equation (2) on 4 sites only.

## 3.3 Duality relations for the porous medium model

We now show how to use the above algebraic approach to find a self-duality relation. In general, it is simpler to have a duality function in a factorized and space homogeneous form, i.e. which can be written in the following way

$$D(\eta, \xi) = \prod_x d(\eta_x, \xi_x) , \tag{39}$$

so that one can focus on finding the single site self-duality function $d(\eta_x, \xi_x)$. However, unlike all the other IPS for which self-dualities have been established, in this model the jump rates depend not only on the state of the two sites involved but also on neighboring sites. This will lead the analysis to two different self-duality functions, one that cannot be factorized unless losing some information, while the other one has a factorized structure but it is non-homogeneous over the lattice site.

### 3.3.1 Duality function I

It is easy to check that on $\mathbb{T}_n$ the PMM has the same reversible measure than the SSEP, i.e. homogeneous product of Bernoulli with free parameter $\rho \in (0,1)$:

$$\mu(\eta : \eta(x) = 1) = \rho = 1 - \mu(\eta : \eta(x) = 0).$$

Therefore, by Proposition 2 a cheap self-duality function is guaranteed to exist. In virtue of the fact that the total number of particles is conserved by the dynamic of the model (recall Remark 9, the state space of the process is $\Omega_n = \{0,1\}^{\mathbb{T}_n}$), then duality functions that differ by constants or quantities that are kept constant by the dynamics (e.g. the total number of particles) are equivalent (see Lemma 3 of [4]). In our case, this means that any term that depends only on $\rho$ can be neglected for the self-duality function, and so we have that the cheap self-duality function is

$$D^{cheap}(\eta,\xi) = \prod_{x\in\mathbb{T}_n} d^{cheap}(\eta_x,\xi_x) = \prod_{x\in\mathbb{T}_n} \delta_{\eta_x,\xi_x} \,. \tag{40}$$

We now look for our symmetry $S$, acting on $D^{cheap}$, to find a non-trivial self-duality function. To do this we start with the result for the SSEP and extend it for the PMM. For the SSEP it is known that the $\mathfrak{su}(2)$ algebra generator $J^+$ is a symmetry of the Casimir by definition of the Casimir element. By Theorem 3 its coproduct is a symmetry of the coproduct of the Casimir which means that we have a symmetry for the SSEP generator. In order to have self-duality in a factorized form, we will consider its exponential, $e^{J^+}$. By inspection of the PMM generator one then sees that for sites 2 and 3, the same symmetry must hold. However, the generator used in sites 1 and 4 do not commute with $J^+$, and so we extend the symmetry using the identity operator. This is formalized in the following lemma.

**Lemma 1.** *In the context of the Lie algebra $\mathfrak{su}(2)$, the following operator*

$$S = 1_1 \otimes e^{J_2^+} \otimes e^{J_3^+} \otimes 1_4$$

*is a symmetry of the generator $L_P$.*

*Proof.* One way to verify this is, for example, to show that $[L_P^{2,3}, S] = 0$. On the other hand, from the expression of $L_P$ in equation (37) the second parenthesis only involves sites 2 and 3 and it commutes with $e^{J_2^+} \otimes e^{J_3^+}$, while for the first parenthesis we just use the fact that the identity operator is always a symmetry.

*Remark 10.* At this point, it is important to stress the following observation. One would expect that, given that the PMM and the SSEP are described via the same algebra, with the only difference of the operator $J^0$ (which does not increase nor decrease the degree of the functions), then a duality relation for the PMM would be close to the SSEP one. However, we can already see from the space non-homogeneity expression of the symmetry $S$ that this cannot be the case.

Following Theorem 1 we now have to act with $S$ on $D^{cheap}$ in order to construct a new non-trivial self-duality function, namely

$$D(\eta,\xi) = SD^{cheap}(\cdot,\xi)(\eta)$$

in operator notation or $D = SD^{cheap}$ in matrix notation. The matrix $S$ can be written as $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which leads to

$$S = \left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & A \end{array}\right),$$

where $A$ is the following lower triangular block:

$$A = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1 \end{pmatrix} .$$

Since $D^{cheap}$ is just the identity matrix, we have that the above matrix is also our new non-trivial self-duality matrix. As a function, we can see that the matrix $D$ above can be written in a factorized form as

$$D(\eta,\xi) = \mathbf{1}_{\{\eta_1=\xi_1\}}\mathbf{1}_{\{\eta_2\geq\xi_2\}}\mathbf{1}_{\{\eta_3\geq\xi_3\}}\mathbf{1}_{\{\eta_4=\xi_4\}} . \tag{41}$$

For the first and the last factors above no computations are needed, while for the second and third it is enough to see that, for the single site self-duality function we have

$$\begin{aligned} d(\eta,\xi) &= e^{J^+} d^{cheap}(\cdot,\xi)(\eta) = \sum_{i=0}^{\infty} \frac{(J^+)^i}{i!}\delta_{\eta,\xi} = \sum_{i=0}^{\infty} \frac{1}{i!}\frac{\eta!}{(\eta-i)!}\mathbf{1}_{\{i\leq\eta\}}\delta_{\eta-i,\xi} \\ &= \binom{\eta}{\xi}\mathbf{1}_{\{0\leq\xi\leq\eta\}} = \mathbf{1}_{\{\xi\leq\eta\}} . \end{aligned}$$

The last equality follows from the fact that we are only dealing with 0 or 1, whose factorial is always 1. As expected, given the observation in Remark 10, the self-duality function of equation (41) is non-homogeneous. If we want to write this in a homogeneous form the only possibility is to "lose" some information in the two central sites. Namely, to impose that the single-site duality function for sites 2 and 3 matches the one for sites 1 and 4 and so losing the choice $\{\eta<\xi\}$. This allows to retrieve a duality function on the torus, which matches the cheap duality function, i.e.

$$D(\eta,\xi) = \prod_{x\in\mathbb{T}_n} \mathbf{1}_{\{\eta_x=\xi_x\}} .$$

It is a bit counter-intuitive that diagonal operators such as the $J^0$, which is morally a multiplicative operator actually do change profoundly the form of the self-duality function compared to the self-duality of the SSEP, see Remark 11 below. We could claim that duality relations are not robustness when kinetically constrains of the microscopic model are taken into consideration.

*Remark 11.* A similar strategy for the SSEP leads to the self-duality function

$$D(\eta,\xi) = \prod_{x\in\mathbb{T}_n} \mathbf{1}_{\{\eta_x\geq\xi_x\}} , \tag{42}$$

which is useful because one can write it in terms of the $\eta$ process. For example assuming $\xi = \delta_x$, the dual configuration with just one particle at site $x \in \mathbb{T}_n$, the expression of the self-duality function in equation (42) reads for all $y \in \mathbb{T}_n - \{x\}$ as $\mathbf{1}_{\{\eta_y \geq 0\}}$ which is always one. While for site $x$ we have

$$\mathbf{1}_{\{\eta_x \geq 1\}} = \begin{cases} 0 & \text{if } \eta_x = 0 \\ 1 & \text{if } \eta_x = 1 \end{cases} = \eta_x$$

leading to $D(\eta, \delta_x) = \eta_x$. A similar reasoning leads to $D(\eta, \delta_x + \delta_y) = \eta_x \eta_y$ and so on. This is how duality relates $n$ dual walkers with the $n-$point correlation function of the original process.

### 3.3.2 Duality function II

We conclude this overview of self-duality for IPS with a different, more direct approach that produces a distinct, far from trivial self-duality function. The idea is to take advantage of the self-duality relation of the SSEP with the self-duality function in equation (42). We have seen that, thanks to the algebraic approach, the PMM and the SSEP generators are connected by

$$L_P = (J_1^0 + J_4^0 - 1)L_S. \tag{43}$$

Looking for a self-duality relation introduced in equation (29), it means that

$$(\eta_1 + \eta_4)L_S D(\cdot, \xi)(\eta) = (\xi_1 + \xi_4)L_S D(\eta, \cdot)(\xi)\ .$$

Now the key observation is that $L_S D(\cdot, \xi)(\eta) = L_S D(\eta, \cdot)(\xi)$ holds for every $D$ of the form we are interested in as in equation (39). Indeed, for two neighboring sites (say 2 and 3 to be consistent) we assume that

$$D(\eta, \xi) = d_2(\eta_2, \xi_2) d_3(\eta_3, \xi_3)\ .$$

Since we are working with the hard-core exclusion we have that for $x = 2, 3$ the possible choices for $d$ are

$$d(\eta_x, \xi_x) = A + B\eta_x + C\xi_x + D\eta_x\xi_x\,,$$

for $A$, $B$, $C$ and $D$ arbitrary constant where $\eta_x, \xi_x \in \{0, 1\}$. At this point, it is not hard to verify via an explicit, long but trivial computation that the duality relation in Definition 5 holds independently from the choices of $A$, $B$, $C$ or $D$. This means that we have the freedom to choose $D$ such that it just has to satisfy the identity $\eta_1 + \eta_4 = \xi_1 + \xi_4$. In other words, we are saying that a self-duality function requires to have the same number of particles in the original and dual process for sites that have distance 2, i.e.

$$D(\eta,\xi)=\prod_{x\in\mathbb{T}_n}\mathbf{1}_{\{\eta_{x-1}+\eta_{x+2}=\xi_{x-1}+\xi_{x+2}\}}\,.$$

If the self-duality function above can be of any interest, it is not clear at this stage. We were able to have a product form which, unluckily, is inhomogeneous; in a symmetric context, all useful applications, as far as we know only deal with space homogeneous self-duality functions.
We conclude here the section regarding self-duality for the PMM. A question, still open, would be to figure out if we can also have an algebraic description of the generators $L_\alpha$ and $L_\beta$ of equation (6). This would give more insight in the form of the dual process at the boundaries. Moreover, we believe that, if the Glauber dynamic allows an algebraic description, one would have hope to infer the one or two points correlation functions using the dual – possibly only absorbing – process.

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## Appendix

Finally, we write some auxiliary results for the work to be consistent. We start by showing that the quadratic variation vanishes in $L^2(\mathbb{P}_{\mu_n})$, as $n$ goes to infinity, and we then recall the replacement lemmas needed to prove the Fick's law.

### *Quadratic variation*

In this section we will prove that the quadratic variation of (22) vanishes in $L^2(\mathbb{P}_{\mu_n})$, as $n$ goes to infinity.
Fix $f\in C^1(0,1)$. From Dynkin's formula (see Lemma A1.5.1 of [18]) we have that

$$M_t^n(f)=J_t^n(f)-J_0^n(f)-\int_0^t n^2\tilde{L}_nJ_s^n(f)\,ds,$$

is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. The quadratic variation of $M_t^n$ is given by $\langle M^n(f)\rangle_t=\int_0^t B_s^n(f)\,ds$, where

$$B_s^n(f):=n^2\left(\tilde{L}_nJ_s^n(f)^2-2J_s^n(f)\tilde{L}_nJ_s^n(f)\right).$$

Recalling the definition of $\tilde{L}_n$ in (13), we can write $B_s^n(f)$ in the following form

$$B_s^n(f)=B_{s,\alpha}^n(f)+B_{s,P}^n(f)+n^{a-2}B_{s,S}^n(f)+B_{s,\beta}^n(f).\tag{44}$$

Let us examine the conservative part of (44). Note that

$$\left(B^n_{s,P}+n^{a-2}B^n_{s,S}\right)(f)=n^2\left(\left(\tilde{L}_P+n^{a-2}\tilde{L}_S\right)Q^n_s(f)^2-2Q^n_s(f)\left(\tilde{L}_P+n^{a-2}\tilde{L}_S\right)Q^n_s(f)\right). \tag{45}$$

To simplify notation, take $Q^n_s(f)=F(\eta_{sn^2},Q^n_s(x))$. Now, we can write (45) as

$$\left(B^n_{s,P}+n^{a-2}B^n_{s,S}\right)(f)=n^2\sum_{x=1}^{n-1}\left(B^n_{s,P}+n^{a-2}B^n_{s,S}\right)(x),$$

where

$$\begin{aligned}\left(B^n_{s,P}+n^{a-2}B^n_{s,S}\right)(x)&=\left(\tilde{L}_P+n^{a-2}\tilde{L}_S\right)F(\eta_{sn^2},Q^n_s(x))^2\\&\quad-2F(\eta_{sn^2},Q^n_s(x))\left(\tilde{L}_P+n^{a-2}\tilde{L}_S\right)F(\eta_{sn^2},Q^n_s(x)).\end{aligned}$$

The previous expression is equal to

$$\begin{aligned}&a_{x,x+1}(\eta_{sn^2})(c_{x,x+1}(\eta_{sn^2})+n^{a-2})\left(F\left(\eta^{x,x+1}_{sn^2},Q^n_s(x)+1\right)-F\left(\eta_{sn^2},Q^n_s(x)\right)\right)^2\\&+a_{x+1,x}(\eta_{sn^2})(c_{x,x+1}(\eta_{sn^2})+n^{a-2})\left(F\left(\eta^{x,x+1}_{sn^2},Q^n_s(x)-1\right)-F\left(\eta_{sn^2},Q^n_s(x)\right)\right)^2\\&+\sum_{\substack{y=1\\y\neq x}}^{n-2}(\eta_{sn^2}(x)-\eta_{sn^2}(x+1))^2(c_{x,x+1}(\eta_{sn^2})+n^{a-2})\left(F\left(\eta^{y,y+1}_{sn^2},Q^n_s(y)\right)-F(\eta_{sn^2},Q^n_s(y))\right)^2.\end{aligned}$$

Thus, since $Q^n_s(f)=F(\eta_{sn^2},Q^n_s(x))$, we get

$$\begin{aligned}&a_{x,x+1}(\eta_{sn^2})(c_{x,x+1}(\eta_{sn^2})+n^{a-2})\left(\frac{1}{n^2}\sum_{y=1}^{n-2}f\left(\tfrac{y}{n}\right)Q^{n,x+1}_s(y)-\frac{1}{n^2}\sum_{y=1}^{n-2}f\left(\tfrac{y}{n}\right)Q^n_s(y)\right)^2\\&+a_{x+1,x}(\eta_{sn^2})(c_{x,x+1}(\eta_{sn^2})+n^{a-2})\left(\frac{1}{n^2}\sum_{y=1}^{n-2}f\left(\tfrac{y}{n}\right)Q^{n,x-1}_s(y)-\frac{1}{n^2}\sum_{y=1}^{n-2}f\left(\tfrac{y}{n}\right)Q^n_s(y)\right)^2,\end{aligned}$$

which is equal to

$$\begin{aligned}&a_{x,x+1}(\eta_{sn^2})(c_{x,x+1}(\eta_{sn^2})+n^{a-2})\left(\frac{1}{n^2}f\left(\tfrac{x}{n}\right)(Q^n_s(x)+1)-\frac{1}{n^2}f\left(\tfrac{x}{n}\right)Q^n_s(x)\right)^2\\&+a_{x+1,x}(\eta_{sn^2})(c_{x,x+1}(\eta_{sn^2})+n^{a-2})\left(\frac{1}{n^2}f\left(\tfrac{x}{n}\right)(Q^n_s(x)-1)-\frac{1}{n^2}f\left(\tfrac{x}{n}\right)Q^n_s(x)\right)^2.\end{aligned}$$

Hence,

$$\left(B^n_{s,P}+n^{a-2}B^n_{s,S}\right)(x)=\frac{1}{n^4}f\left(\tfrac{x}{n}\right)^2(a_{x,x+1}(\eta_{sn^2})+a_{x+1,x}(\eta_{sn^2}))(c_{x,x+1}(\eta_{sn^2})+n^{a-2}).$$

Therefore,

$$
\begin{aligned}
\left(B^n_{s,P}+n^{a-2}B^n_{s,S}\right)(f) &= \frac{1}{n^2}\sum_{x=1}^{n-2} f\left(\tfrac{x}{n}\right)^2 (a_{x,x+1}(\eta_{sn^2})+a_{x+1,x}(\eta_{sn^2}))(c_{x,x+1}(\eta_{sn^2})+n^{a-2})\\
&\leq 2\frac{\|f^2\|_\infty}{n}+\frac{\|f^2\|_\infty}{n^{3-a}},
\end{aligned}
$$

which vanishes when $n$ goes to infinity since $1<a<2$.
Let us now examine the non-conservative part of the quadratic variation. Note that

$$
\left(B^n_{s,\alpha}+B^n_{s,\beta}\right)(f)=n^2\left(\left(\tilde{L}_\alpha+\tilde{L}_\beta\right)K^n_s(f)^2-2K^n_s(f)\left(\tilde{L}_\alpha+\tilde{L}_\beta\right)K^n_s(f)\right).
$$

We will examine only $B^n_{s,\alpha}(f)$ since the computations for $B^n_{s,\beta}(f)$ are the same. Take $K^n_s(f)=F(\eta_{sn^2},K^n_s(0))$. Repeating the same arguments used above, we have that

$$
\begin{aligned}
B^n_{s,\alpha}(f)=n^2\frac{m}{n^\theta}\Bigg\{&\alpha(1-\eta_{sn^2}(1))\left(F\left((\eta^1_{sn^2},K^n_s(0)+1\right)-F(\eta_{sn^2},K^n_s(0))\right)^2\\
&+(1-\alpha)(\eta_{sn^2}(1))\left(F\left((\eta^1_{sn^2},K^n_s(0)-1\right)-F(\eta_{sn^2},K^n_s(0))\right)^2\Bigg\}.
\end{aligned}
$$

Since $K^n_s(f)=F(\eta_{sn^2},K^n_s(0))$, we get

$$
\begin{aligned}
n^2\frac{m}{n^\theta}\Bigg\{&\alpha(1-\eta_{sn^2}(1))\left(\frac{1}{n}f(0)(K^n_s(0)+1)-\frac{1}{n}f(0)K^n_s(0)\right)^2\\
&+(1-\alpha)(\eta_{sn^2}(1))\left(\frac{1}{n}f(0)(K^n_s(0)-1)-\frac{1}{n}f(0)K^n_s(0)\right)^2\Bigg\}.
\end{aligned}
$$

Hence,

$$
B^n_{s,\alpha}(f)=\frac{m}{n^\theta}f(0)^2(\alpha-\eta_{sn^2}(1))^2.
$$

In the same manner, we also have

$$
B^n_{s,\beta}(f)=\frac{m}{n^\theta}f\left(\tfrac{n-1}{n}\right)^2(\beta-\eta_{sn^2}(n-1))^2.
$$

Therefore,

$$
B^n_{s,\alpha}(f)+B^n_{s,\beta}(f)\leq C(\alpha,\beta)\frac{m}{n^\theta}\|f^2\|_\infty, \tag{46}
$$

which vanishes as $n$ goes to infinity for any $\theta>0$. In order to conclude the proof we need to show that (46) vanishes for $\theta=0$. This case is proved in Proposition 4.1 of [7] and we refer the interested reader to see the proof there.

## *Replacement lemmas*

In this section, we state all the replacement lemmas used along the paper. For the proofs, we refer the reader to [3]. Before stating the results let us fix some notation. Fix $n,\ell \in \mathbb{N}$, $x \in \Sigma_n$ and $\varepsilon > 0$. Let $\Sigma_n^\varepsilon = \{1+\varepsilon n, \dots, n-1-\varepsilon n\}$, where $\varepsilon n$ denotes $\lfloor \varepsilon n \rfloor$,

$$\overleftarrow{\Lambda}_x^\ell := \{x-\ell+1,\dots,x\} \quad \text{and} \quad \overrightarrow{\Lambda}_x^\ell := \{x,\dots,x+\ell-1\},$$

be the boxes of size $\ell$ to the left and to the right of site $x$, respectively. We denote by

$$\overleftarrow{\eta}^\ell(x) = \frac{1}{\ell}\sum_{y\in\overleftarrow{\Lambda}_x^\ell}\eta(y) \quad \text{and} \quad \overrightarrow{\eta}^\ell(x) = \frac{1}{\ell}\sum_{y\in\overrightarrow{\Lambda}_x^\ell}\eta(y)$$

the empirical densities in the boxes $\overleftarrow{\Lambda}_x^\ell$ and $\overrightarrow{\Lambda}_x^\ell$.

**Theorem 3.** *Let $H:[0,1]\to\mathbb{R}$ be such that $\|H\|_\infty \le M < \infty$. For any $t\in[0,T]$, we have that*

$$\lim_{\varepsilon\to 0}\varlimsup_{n\to+\infty}\mathbb{E}_{\mu_n}\left(\int_0^t \frac{1}{n}\sum_{x\in\Sigma_n^\varepsilon} H\left(\tfrac{x}{n}\right)\left(\eta_{sn^2}(x)\eta_{sn^2}(x+1) - \overleftarrow{\eta}_{sn^2}^{\varepsilon n}(x)\overrightarrow{\eta}_{sn^2}^{\varepsilon n}(x+1)\right)ds\right) = 0.$$

**Theorem 4.** *For any $t\in[0,T]$, we have*

$$\lim_{\varepsilon\to 0}\varlimsup_{n\to+\infty}\mathbb{E}_{\mu_n}\left(\left|\int_0^t \eta_{sn^2}(1)\eta_{sn^2}(2) - \overrightarrow{\eta}_{sn^2}^{\varepsilon n}(1)\overrightarrow{\eta}_{sn^2}^{\varepsilon n}(\varepsilon n+1)\,ds\right|\right) = 0$$

*and*

$$\lim_{\varepsilon\to 0}\varlimsup_{n\to+\infty}\mathbb{E}_{\mu_n}\left(\left|\int_0^t \eta_{sn^2}(n-1)\eta_{sn^2}(n-2) - \overleftarrow{\eta}_{sn^2}^{\varepsilon n}(n-1)\overleftarrow{\eta}_{sn^2}^{\varepsilon n}(n-1-\varepsilon n)\,ds\right|\right) = 0.$$

**Theorem 5.** *Fix $\theta < 1$. Let $\varphi:\Omega_n\to\Omega_n$ be a positive and bounded function which does not depend on the value of the configuration $\eta$ at site 1. For any $t\in[0,T]$, we have that*

$$\lim_{\varepsilon\to 0}\varlimsup_{n\to+\infty}\mathbb{E}_{\mu_n}\left(\left|\int_0^t \varphi(\eta_{sn^2})(\alpha-\eta_{sn^2}(1))\,ds\right|\right) = 0.$$

*The same is true for $\beta$ if place of $\alpha$, $n-1$ in place of 1 and requiring $\varphi$ not to depend on $\eta$ at site $n-1$.*

**Theorem 6.** *For any $t\in[0,T]$, we have*

$$\lim_{\varepsilon\to 0}\varlimsup_{n\to+\infty}\mathbb{E}_{\mu_n}\left(\left|\int_0^t \eta_{sn^2}(1) - \overrightarrow{\eta}_{sn^2}^{\varepsilon n}(1)\,ds\right|\right) = 0,$$

*The same is true for $\eta_{sn^2}(n-1)$ if place of $\eta_{sn^2}(1)$ and $\overleftarrow{\eta}_{sn^2}^{\varepsilon n}(n-1)$ in place of $\overrightarrow{\eta}_{sn^2}^{\varepsilon n}(1)$.*

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